# ON THE USE OF SOMOFF'S THEOREM FOR THE EVALUATION OF THE ELLIPTIC INTEGRAL OF THE THIRD SPECIES.

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[continued from vol. II, page 77.]

For the convenient computation of the elements of the next higher step, I propose the following algorithm. Starting with $a = 1$, $b = \beta$, $c = \gamma$, $a\Delta\varphi$, $c\cos\varphi$, $a\Delta\mu$, $c\cos\mu$, as data, compute as follows :—

$$\begin{aligned}
a' &= \tfrac{1}{2}(a + c), & b' &= \tfrac{1}{2}(a - c), \quad c' = \sqrt{(ac)}\,; \\
a'\Delta\varphi' &= \tfrac{1}{2}(a\Delta\varphi + c\cos\varphi), & c'\cos\varphi' &= \sqrt{(a'\Delta\varphi' + b')}\,.\,\sqrt{(a'\Delta\varphi' - b')}, \\
a'\Delta\mu' &= \tfrac{1}{2}(a\Delta\mu + c\cos\mu), & c'\cos\mu' &= \sqrt{(a'\Delta\mu' + b')}\,.\,\sqrt{(a'\Delta\mu' - b')}, \\
a'\Delta\mu^{\backprime} &= \tfrac{1}{2}(a\Delta\mu - c\cos\mu), & c'\cos\mu^{\backprime} &= \sqrt{(a'\Delta\mu^{\backprime} + b')}\,.\,\sqrt{(a'\Delta\mu^{\backprime} - b')}. \quad (18)
\end{aligned}$$

We thus obtain the new elements of the next higher step in the modular scale, viz. $\gamma' = c' / a'$, $\varphi'$, $\mu'$, $\mu^{\backprime}$, from which (16) can be formed. Computing in the same manner $a''$, $b''$, $c''$ from $a'$, $b'$, $c'$ ; $a''\Delta\varphi''$ and $c''\cos\varphi''$ from $a'\Delta\varphi'$ and $c'\cos\varphi'$ ; $a''\Delta\mu''$ and $c''\cos\mu''$, as well as $a''\Delta\mu'^{\backprime}$ and $c''\cos\mu'^{\backprime}$, from $a'\Delta\mu'$ and $c'\cos\mu'$; $a''\Delta\mu^{\backprime\prime}$ and $c''\cos\mu^{\backprime\prime}$, as well as $a''\Delta\mu^{\backprime\backprime}$ and $c''\cos\mu^{\backprime\backprime}$, from $a'\Delta\mu^{\backprime}$ and $c'\cos\mu^{\backprime}$, we obtain the elements of the equation

$$\Pi(\varphi_\gamma, \mu_\gamma) = \Pi(\varphi''_{\gamma''}, \mu''_{\gamma''}) - \Pi(\varphi''_{\gamma''}, \mu'^{\backprime}_{\gamma''}) - \Pi(\varphi''_{\gamma''}, \mu^{\backprime\prime}_{\gamma''}) + \Pi(\varphi''_{\gamma''}, \mu^{\backprime\backprime}_{\gamma''}). \quad (16'')$$

Continuing this process until $a^{(n)} = c^{(n)} =$ arithmetico-geometric mean of $a$ and $c$, and also $a^{(n)}\Delta\varphi^{(n)} = c^{(n)}\cos\varphi^{(n)}$, $a^{(n)}\Delta\mu^{(n-1)'} = c^{(n)}\cos\mu^{(n-1)'}$, but $\Delta\mu^{(n-1)'} = \cos\mu^{(n-1)\backprime} = 0$, then we have finally

$$\Pi(\varphi_\gamma, \mu_\gamma) = \Sigma\,[(-)^s\,\Pi\,(\varphi^{(n)}_{\gamma^{(n)}}, \mu^{(n-1)'}_{\gamma^{(n)}})] = \Sigma\,[(-)^s\,\Pi\,(\varphi^{(n)}_1, \mu^{(n-1)'}_1)], \quad (16^{(n)})$$

where $s$ denotes the number of grave accents affixed to $\mu$.

The integral is now expressed in $2^n$ terms of the same form ; viz. :—

$$\begin{aligned}
\Pi\,(\varphi^{(n)}_1, \mu^{(n-1)'}_1) &= \int_0^{\phi^{(n)}} \sin\mu^{(n-1)'}\cos^2\mu^{(m-1)'}\,.\,\frac{d\varphi}{\cos\varphi}\,.\,\frac{\sin^2.\varphi}{1 - \sin^2\mu^{(n-1)'}\sin^2\varphi} \\
&= \tfrac{1}{2}\sin\mu^{(n-1)'}\, l\,\frac{1 + \sin\varphi^{(n)}}{1 - \sin\varphi^{(n)}} - \tfrac{1}{2}\, l\,\frac{1 + \sin\mu^{(n-1)'}\sin\varphi^{(n)}}{1 - \sin\mu^{(n-1)'}\sin\varphi^{(n)}}. \quad (19)
\end{aligned}$$

This is the standard analytical form. To adapt it to the use of logarithmic tables, place

$$\cos\lambda^{(n-1)'} = \sin\mu^{(n-1)'}\sin\varphi^{(n)}\,; \quad (20)$$

then, since $$\varphi_\gamma = \frac{1}{2a^{(n)}}\, l\,\frac{1 + \sin\varphi^{(n)}}{1 - \sin\varphi^{(n)}} = \frac{1}{c^{(n)}}\, l\tan\tfrac{1}{2}(\lrcorner + \varphi^{(n)}), \quad (21)$$

we have

$$\Pi(\varphi_1^{(n)}, \mu_1^{(n-1)'}) \log e = \sin \mu^{(n-1)'} \log \tan \tfrac{1}{2} (\lrcorner + \varphi^{(n)}) - \log \cot \tfrac{1}{2}\lambda^{(n-1)'}$$
$$= c^{(n)} \sin \mu^{(n-1)'} \varphi_\gamma \log e - \log \cot \tfrac{1}{2}\lambda^{(n-1)'}. \qquad (19')$$

Computing this quantity for each limit, and combining according to ($16^{(n)}$), we obtain $\Pi(\varphi_\gamma, \mu_\gamma) \log e$, whence $\Pi(\varphi_\gamma, \mu_\gamma)$.

This method is not applicable to the cyclometric integral $\Pi(\varphi_\gamma, \nu_\beta i)$, as I had at first thought, although the algorithm (18) is applicable to $a\,\Delta(\nu_\beta i)_{-\gamma} = a\,\Delta\nu$ sec $\nu$, $c \cos(\nu_\beta i)_{-\gamma} = c \sec \nu$ in a straight line, because these and $a'\Delta(\nu'_{\beta'} i)_{-\gamma'}$, $c' \cos(\nu'_{\beta'} i)_{-\gamma'}$ and $a''\Delta(\nu''_{\beta''} i)_{-\gamma''}$, $c'' \cos(\nu''_{\beta''} i)_{-\gamma''}$, etc., are all real. But, although $a'\Delta(\nu'_{\beta'} i)_{-\gamma'}$ is also real, $c' \cos(\nu'_{\beta'} i)_{-\gamma'} = \sqrt{[a'^2\Delta^2(\nu'_{\beta'} i)_{-\gamma'} - b'^2]}$ is not, because $a'\Delta(\nu'_{\beta'} i)_{-\gamma'} < b'$. This makes the above method, which I shall distinguish as the first method, impracticable for the cyclometric integral, at least for the ascending scale.

There is a slight defect in this method in case $\varphi$ is small; for the algorithm (18) ends with $a^{(n)}\,\Delta\varphi^{(n)} = c^{(n)} \cos \varphi^{(n)}$, or $\Delta\varphi^{(n)} = \cos \varphi^{(n)}$, which is near unity if $\varphi$, and hence $\varphi^{(n)}$, are small. The determination of $\varphi^{(n)}$ from the cosine is in that case not precise.

Now, since
$$\sin \varphi = (1 + \beta') \frac{\sin \varphi' \cos \varphi'}{\Delta\varphi'}$$
$$= \frac{a \sin \varphi' \cos \varphi'}{a'\Delta\varphi'},$$

we have
$$c \sin \varphi = \frac{c' \sin \varphi' . c' \cos \varphi'}{a'\Delta\varphi'};$$
$$\therefore\ c' \sin \varphi' = \frac{a'\Delta\varphi'}{c' \cos \varphi'} . c \sin \varphi. \qquad (22)$$

Adding this easy computation for each step of the modular scale, we have finally both $c^{(n)} \sin \varphi^{(n)}$ and $c^{(n)} \cos \varphi^{(n)}$ from the algorithm, and $\varphi^{(n)}$ can be determined from its sine, cosine, or tangent. Moreover, since (20) requires $\sin \varphi^{(n)}$ and $\sin \mu^{(n)}$, this results directly.

While this controlling computation is not essential to this method, it is to the second method, which is based on form (17) of Somoff's theorem. We have, at the second step of the scale,

$$\Pi(\varphi_\gamma, \mu_\gamma) = 2^2\, \Pi(\varphi''_{\gamma''}, \mu''_{\gamma''}) - 2\left(\gamma' \sin \mu' \varphi'_{\gamma'} - \tfrac{1}{2}l \frac{1 + \gamma' \sin \mu' \sin \varphi'}{1 - \gamma' \sin \mu' \sin \varphi'}\right)$$
$$- \gamma \sin \mu\, \varphi_\gamma + \tfrac{1}{2}l \frac{1 + \gamma \sin \mu \sin \varphi}{1 - \gamma \sin \mu \sin \varphi}$$

$$= 2^2 \Pi(\varphi''_{\gamma''}, \mu''_{\gamma''}) - (2c' \sin \mu' + c \sin \mu)\, \varphi_\gamma + l \cot^2 \tfrac{1}{2}\lambda' \cot \tfrac{1}{2}\lambda, \tag{23}$$

where

$$\cos \lambda^{(0)} = \gamma \sin \mu \sin \varphi = \frac{c \sin \mu \,.\, c \sin \varphi}{a \,.\, c} \tag{24}$$

$$\cos \lambda' = \gamma' \sin \mu' \sin \varphi' = \frac{c' \sin \mu' .\, c' \sin \varphi'}{a' .\, c'}; \tag{24'}$$

and finally, at the limit,

$$\Pi(\varphi_\gamma, \mu_\gamma) = 2^n \Pi(\varphi_1^{(n)}, \mu_1^{(n)}) - \varphi_\gamma (2^{n-1} c^{(n-1)} \sin \mu^{(n-1)} + \ldots + 2c' \sin \mu' + c \sin \mu)$$
$$+ \; 2^{n-1} l \cot \tfrac{1}{2}\lambda^{(n-1)} + \ldots + 2l \cot \tfrac{1}{2}\lambda' + l \cot \tfrac{1}{2}\lambda^{(0)}. \tag{25}$$

The first term is, of course, computed by (19'). If, however, the control (22) is used, the computation of the limiting step may be omitted; for, since then

$$a^{(n)} \Delta\varphi^{(n)} = c^{(n)} \cos \varphi^{(n)},$$

we have by (22)

$$c^{(n)} \sin \varphi^{(n)} = c^{(n-1)} \sin \varphi^{(n-1)},$$

$$c^{(n)} \sin \mu^{(n)} = c^{(n-1)} \sin \mu^{(n-1)};$$

therefore

$$\cos \lambda^{(n)} = \cos \lambda^{(n-1)},$$

and

$$\Pi(\varphi_1^{(n)}, \mu_1^{(n)}) = c^{(n)} \sin \mu^{(n)} \varphi_\gamma - l \cot \tfrac{1}{2} \lambda^{(n)}$$
$$= c^{(n-1)} \sin \mu^{(n-1)} \varphi_\gamma - l \cot \tfrac{1}{2} \lambda^{(n-1)}.$$

Regarding this, we have, instead of (25),

$$\Pi(\varphi_\gamma, \mu_\gamma)$$
$$= \varphi_\gamma (2^{n-1} c^{(n-1)} \sin \mu^{(n-1)} - 2^{n-2} c^{(n-2)} \sin \mu^{(n-2)} - \ldots - [2c' \sin \mu' - c \sin \mu)$$
$$- 2^{(n-1)} l \cot \tfrac{1}{2} \lambda^{(n-1)} + 2^{(n-2)} l \cot \tfrac{1}{2} \lambda^{(n-2)} + \ldots + 2\, l \cot \tfrac{1}{2} \lambda' + l \cot \tfrac{1}{2} \lambda. \tag{25'}$$

Now, with a slight modification, this method may be used for the cyclometric integral. Placing, for the sake of uniformity,

$$\mu_\gamma = \nu_\beta i, \tag{26}$$

then the algorithm for parameter starts with

$$a\, \Delta\mu = a\, \Delta(\nu_\beta i)_{-\gamma} = a\, \Delta\nu \sec \nu \tag{27}$$

and

$$c \cos \mu = c \cos (\nu_\beta i)_{-\gamma} = c \sec \nu, \tag{28}$$

and is real in a straight line. The control (22) becomes

$$c' \sin \mu' = c' \sin (\nu'_{\beta'} i)_{-\gamma'} = \frac{a' \Delta\mu'}{c' \cos \mu'} .\, c \sin \mu = ic' \tan \nu',$$

or

$$c' \tan \nu' = \frac{a' \Delta\mu'}{c' \cos \mu'} .\, c \tan \nu. \tag{29}$$

The formula (24) cannot be used, since $\lambda$ is not real. Placing, instead,

$$\tan x^{(0)} = \frac{c \tan \nu \, . \, c \sin \varphi}{a \, . \, c}, \tag{30}$$

we have

$$\tfrac{1}{2} l \frac{1 + \gamma \sin \mu \sin \varphi}{1 - \gamma \sin \mu \sin \varphi} = \tfrac{1}{2} l \frac{1 + \gamma i \tan \nu \sin \varphi}{1 - \gamma i \tan \nu \sin \varphi}$$

$$= \tfrac{1}{2} l \frac{1 + i \tan x^{(0)}}{1 - i \tan x^{(0)}}$$

$$= i x^{(0)}, \tag{31}$$

and therefore

$$\Pi (\varphi_\gamma, \nu_\beta i)$$

$$= \varphi_\gamma i \left(2^{n-1} c^{(n-1)} \tan \nu^{(n-1)} - 2^{n-2} c^{(n-2)} \tan \nu^{(n-2)} - \ldots - 2c' \tan \nu' - c \tan \nu\right)$$

$$- i \left(2^{n-1} x^{(n-1)} - 2^{n-2} x^{(n-2)} - \ldots - 2x' - x^{(0)}\right). \tag{32}$$

There is, however, a modification of these methods required, if the modulus $\gamma$ is very small; for, although even then the method necessarily converges, it will do so after a greater number of steps have been ascended, which makes the computation both tedious and less precise. If $\gamma$ is small then $\beta$ is near unity. The required modification is therefore obtained by reducing the integral to the complementary modulus by Jacobi's imaginary transformation. We have thus

$$\Pi (\varphi_\gamma, \mu_\gamma) = \Pi (\psi_\beta i, \nu_\beta i)$$

$$= \int_0^{\psi_\beta i} \sin (\nu_\beta i)_{-\gamma} \cos (\nu_\beta i)_{-\gamma} \Delta (\nu_\beta i)_{-\gamma} \cdot \frac{d\varphi}{\Delta\varphi} \cdot \frac{\gamma^2 \sin \varphi}{1 - \gamma^2 \sin^2 (\nu_\beta i)_{-\gamma} \sin^2 \varphi}$$

$$= \int_0^{\psi} i \tan \nu \frac{\Delta (\nu_\beta)_{-\beta}}{\cos^2 \nu} \cdot i \frac{d\psi}{\Delta (\psi_\beta)_{-\beta}} \cdot \frac{-\gamma^2 \tan^2 \psi}{1 - \gamma^2 \tan^2 \nu \tan^2 \psi}$$

$$= \int_0^{\psi} \tan \nu \frac{\Delta\nu}{\cos^2 \nu} \cdot \frac{d\psi}{\Delta\psi} \cdot \frac{\gamma^2 \sin^2 \psi}{1 - \dfrac{\Delta^2 \nu}{\cos^2 \nu} \sin^2 \psi}. \tag{33}$$

Since

$$\sin (\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} = \frac{\Delta\nu}{\beta \cos \nu},$$

$$\cos (\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} = - \frac{i\gamma}{\beta \cos \nu},$$

$$\Delta (\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} = i\gamma \tan \nu;$$

we have

$$\beta^2 \sin (\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} \cos (\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} \Delta (\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta}$$

$$=\gamma^2 \tan\nu \frac{\Delta\nu}{\cos^2\nu},$$

and therefore

$$\Pi(\varphi_\gamma, \mu_\gamma)$$

$$=\int_0^\psi \left\{ \begin{array}{c} \sin(\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} \cos(\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} \Delta(\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} \\ \times \dfrac{d\psi}{\Delta\psi} \cdot \dfrac{\beta^2 \sin^2\psi}{1 - \beta^2 \sin^2(\lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)_{-\beta} \sin^2\psi} \end{array} \right\}$$

$$= \Pi(\psi_\beta, \lrcorner_\beta + \lrcorner_\gamma i + \nu_\beta)$$

$$= \Pi(\psi_\beta, \nu_\beta) - \psi_\beta \tan\nu\, \Delta\nu + \tfrac{1}{2}\, l\, \frac{\cos(\nu_\beta - \psi_\beta)_{-\beta}}{\cos(\nu_\beta + \psi_\beta)_{-\beta}}, \text{ by (9)},$$

$$= \Pi(\psi_\beta, \nu_\beta) - \frac{\varphi_\gamma}{i} \cdot \frac{\tan\mu\, \Delta\mu}{i} + \tfrac{1}{2}\, l\, \frac{1 + \frac{1}{i}\tan\mu\, \Delta\mu \,.\, \frac{1}{i}\tan\varphi\, \Delta\varphi}{1 - \frac{1}{i}\tan\mu\, \Delta\mu \,.\, \frac{1}{i}\tan\varphi\, \Delta\varphi}$$

$$= \Pi(\psi_\beta, \nu_\beta) + \varphi_\gamma \tan\mu\, \Delta\mu - \tfrac{1}{2}\, l\, \frac{\cos(\mu_\gamma - \varphi_\gamma)_{-\gamma}}{\cos(\mu_\gamma + \varphi_\gamma)_{-\gamma}} \tag{34}$$

$$= \Pi(\psi_\beta, \nu_\beta) + \varphi_\gamma \tan\mu\, \Delta\mu - l \cot\tfrac{1}{2}\omega, \text{ if } \cos\omega = \tan\mu\, \Delta\mu \tan\varphi\, \Delta\varphi. \tag{35}$$

The term $\Pi(\psi_\beta, \nu_\beta)$ is treated like a logarithmic integral to the modulus $\beta$. We start the algorithm (18) and control (22) with the data

$$a = 1, \qquad c = \gamma, \qquad b = \beta;$$

$$a\,\Delta\psi = a\,\Delta\varphi \sec\varphi,\ b\cos\psi = b\sec\varphi,\ b\sin\psi = \frac{b}{i}\tan\varphi;$$

$$a\,\Delta\nu = a\,\Delta\mu \sec\mu,\ b\cos\nu = b\sec\mu,\ b\sin\nu = \frac{b}{i}\tan\mu; \tag{36}$$

and computing the next higher step of $\beta$, being the next lower step to $\gamma$, the quantities

$$a_{,} = \tfrac{1}{2}(a + b),\ c_{,} = \tfrac{1}{2}(a - b),\ b_{,} = \sqrt{(ab)};$$

$$a_{,}\,\Delta\psi_{,} = \tfrac{1}{2}(a\,\Delta\psi + b\cos\psi) = a\,\Delta\varphi_{,} \sec\varphi_{,},$$

$$b_{,}\cos\psi_{,} = \sqrt{(a_{,}^2 \Delta^2\psi_{,} - c_{,}^2)} = b_{,}\sec\varphi_{,},$$

$$b_{,}\sin\psi_{,} = \frac{a_{,}\,\Delta\psi_{,}}{b_{,}\cos\psi_{,}} \,.\, b\sin\psi = \frac{b_{,}}{i}\tan\varphi_{,};$$

$$a_{,}\,\Delta\nu_{,} = \tfrac{1}{2}(a\,\Delta\nu + b\cos\nu) = a\,\Delta\mu_{,} \sec\mu_{,},$$

$$b_{,}\cos\nu_{,} = \sqrt{(a_{,}^2 \Delta^2\nu_{,} - c_{,}^2)} = b_{,}\sec\mu_{,},$$

$$b_{,}\sin\nu_{,} = \frac{a_{,}\,\Delta\nu_{,}}{b_{,}\cos\nu_{,}} \,.\, b\sin\nu = \frac{b_{,}}{i}\tan\mu_{,}. \tag{36'}$$

Continuing in this manner until $a_{(n)} = b_{(n)} =$ arithmetico-geometric mean of $a$ and $b$, we have also $a_{(n)}\Delta\psi_{(n)} = b_{(n)}\cos\psi_{(n)}$ or $\Delta\varphi_{(n)}\sec\varphi_{(n)} = \sec\varphi_{(n)}$, and similarly for $\mu$. By (25′) we have now, going back to its more fundamental form,

$$\Pi(\psi_\beta, \nu_\beta)$$

$$= \psi_\beta (2^{n-1} b_{(n-1)} \sin \nu_{(n-1)} - 2^{n-2} b_{(n-2)} \sin \nu_{(n-2)} - \ldots - 2b_{,} \sin \nu_{,} - b \sin \nu)$$

$$- 2^{n-2} l \frac{1 + \beta_{(n-1)} \sin \nu_{(n-1)} \sin \psi_{(n-1)}}{1 - \beta_{(n-1)} \sin \nu_{(n-1)} \sin \psi_{(n-1)}}$$

$$+ 2^{n-3} l \frac{1 + \beta_{(n-2)} \sin \nu_{(n-2)} \sin \psi_{(n-2)}}{1 - \beta_{(n-2)} \sin \nu_{(n-2)} \sin \psi_{(n-2)}}$$

$$+ \ldots$$

$$+ l \frac{1 + \beta_{,} \sin \nu_{,} \sin \psi_{,}}{1 - \beta_{,} \sin \nu_{,} \sin \psi_{,}} + \tfrac{1}{2} l \frac{1 + \beta \sin \nu \sin \psi}{1 - \beta \sin \nu \sin \psi}$$

$$= - \varphi_\gamma (2^{n-1} b_{(n-1)} \tan \mu_{(n-1)} - 2^{n-2} b_{(n-2)} \tan \mu_{(n-2)} - \ldots - 2b_{,} \tan \mu_{,} - b \tan \mu)$$

$$+ 2^{n-1} l \cot \tfrac{1}{2}\lambda_{(n-1)} - 2^{n-2} l \cot \tfrac{1}{2}\lambda_{(n-2)} - \ldots - 2l \cot \tfrac{1}{2}\lambda_{,} - l \cot \tfrac{1}{2}\lambda_{(0)}, \quad (37)$$

where we have placed

$$\cos \lambda_{(0)} = \beta \tan \mu \tan \varphi = \frac{b \tan \mu \,.\, b \tan \varphi}{a \,.\, b},$$

$$\cos \lambda_{,} = \beta_{,} \tan \mu_{,} \tan \varphi_{,} = \frac{b_{,} \tan \mu_{,} \,.\, b_{,} \tan \varphi_{,}}{a_{,} \,.\, b_{,}}, \text{ etc.} \quad (38)$$

We have also here

$$\varphi_\gamma = i\psi_\beta = \frac{i}{2b_{(n)}} l \frac{1 + \sin \psi_{(n)}}{1 - \sin \psi_{(n)}}$$

$$= \frac{\varphi_{(n)}}{b_{(n)}}, \quad (39)$$

and we recognize in this the elegant formula of Gauss for the integral of the first species, and $\varphi_{(n)}$ is the argument of Jacobi's $\vartheta$ functions, usually denoted by $x$.

Treating the cyclometric integral in a similar manner, we have

$$\Pi(\varphi_\gamma, \nu_\beta i) = \Pi(\psi_\beta i, - \mu_\gamma)$$

$$= \Pi(\psi_\beta, \daleth_\beta + \daleth_\gamma i - \mu_\gamma i)$$

$$= \Pi(\psi_\beta, \daleth_\beta + \daleth_\gamma i - \nu_\beta)$$

$$= \Pi(\psi_\beta, - \nu_\beta) + \psi_\beta \tan \nu \,\Delta\nu - \tfrac{1}{2} l \frac{\cos(\nu_\beta - \psi_\beta)_{-\beta}}{\cos(\nu_\beta + \psi_\beta)_{-\beta}}$$

$$= - \Pi(\psi_\beta, \nu_\beta) - i[\varphi_\gamma \tan \nu \,\Delta\nu - \text{arc tan}(\tan \nu \,\Delta\nu \tan \varphi \,\Delta\varphi)]$$

$$= - \Pi(\psi_\beta, \nu_\beta) - i[\varphi_\gamma \tan \nu \,\Delta\nu - o), \text{ if } \tan o = \tan \nu \,\Delta\nu \tan \varphi \,\Delta\varphi. \quad (40)$$

For the first term the algorithm starts with the data

$$a = 1, \qquad c = \gamma, \qquad b = \beta;$$
$$a\,\Delta\psi = a\,\Delta\varphi \sec\varphi, \quad b\cos\psi = b\sec\varphi, \quad b\sin\psi = \frac{b}{i}\tan\varphi;$$
$$a\,\Delta\nu, \qquad b\cos\nu, \qquad b\sin\nu; \tag{41}$$

and, since $\nu$ is real, we can pursue the algorithm for $\nu$ also sideways, by computing also $a_{\prime}\,\Delta\nu_{\prime} = \frac{1}{2}(a\,\Delta\nu + b\cos\nu)$ and following up each branch to its limit. We have then, according to the first method,

$$\Pi(\psi_\beta, \nu_\beta) = \Sigma\left[(-)^s\,\Pi([\psi_{(n)}]_1, [\nu_{(n-1)\prime}]_1)\right], \tag{42}$$

where

$$\Pi([\psi_{(n)}]_1, [\nu_{(n-1)}]_1) = b_{(n)}\sin\nu_{(n-1)\prime}\,\psi_\beta - \tfrac{1}{2}l\,\frac{1 + \sin\nu_{(n-1)\prime}\sin\psi_{(n)}}{1 - \sin\nu_{(n-1)\prime}\sin\psi_{(n)}}$$
$$= -\,b_{(n)}\,i\sin\nu_{(n-1)\prime}\,\varphi_\gamma + i\,\mathrm{arc}\tan(\sin\nu_{(n-1)\prime}\tan\varphi_{(n)})$$
$$= -\,i\,(b_{(n)}\sin\nu_{(n-1)\prime}\,\varphi_\gamma - \chi_{(n-1)\prime}),$$

if
$$\tan\chi_{(n-1)\prime} = \sin\nu_{(n-1)\prime}\tan\varphi_{(n)}. \tag{43}$$

By the second method we have

$$\Pi(\psi_\beta, \nu_\beta)$$
$$= \psi_\beta\,(2^{n-1}b_{(n-1)}\sin\nu_{(n-1)} - 2^{n-2}b_{(n-2)}\sin\nu_{(n-2)} - \ldots - 2b_{\prime}\sin\nu_{\prime} - b\sin\nu)$$
$$- 2^{n-2}\,l\,\frac{1 + \beta_{(n-1)}\sin\nu_{(n-1)}\sin\psi_{(n-1)}}{1 - \beta_{(n-1)}\sin\nu_{(n-1)}\sin\psi_{(n-1)}}$$
$$+ 2^{n-3}\,l\,\frac{1 + \beta_{(n-2)}\sin\nu_{(n-2)}\sin\psi_{(n-2)}}{1 - \beta_{(n-2)}\sin\nu_{(n-2)}\sin\psi_{(n-2)}}$$
$$+ \ldots$$
$$+ l\,\frac{1 + \beta_{\prime}\sin\nu_{\prime}\sin\psi_{\prime}}{1 - \beta_{\prime}\sin\nu_{\prime}\sin\psi_{\prime}} + \tfrac{1}{2}\,l\,\frac{1 + \beta\sin\nu\sin\psi}{1 - \beta\sin\nu\sin\psi}$$
$$= -\,\varphi_\gamma\,i\,(2^{n-1}b_{(n-1)}\sin\nu_{(n-1)} - 2^{n-2}b_{(n-2)}\sin\nu_{(n-2)} - \ldots - 2b_{\prime}\sin\nu_{\prime} - b\sin\nu)$$
$$+ i\,(2^{n-1}\chi_{(n-1)} - 2^{n-2}\chi_{(n-2)} - \ldots - 2\chi_{\prime} - \chi_{(0)}), \tag{44}$$

where
$$\tan\chi_{(s)} = \beta_{(s)}\sin\nu_{(s)}\tan\varphi_{(s)} = \frac{b_{(s)}\sin\nu_{(s)}\,.\,b_{(s)}\tan\varphi_{(s)}}{a_{(s)}\,.\,b_{(s)}}. \tag{45}$$

I have thus shown that the algorithm which forms the main part of the computation is applicable in all cases of the integral of the third species which need to be considered. The practical use of the methods will be shown in a future article.